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ON THE BEST MATCH OF A BILINEAR AND BIQUADRATIC DIGITAL FILTER

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Abstract. A method for computing the coefficients of a second order digital biquadratic filter is presented, matching as close as possible the characteristics of the analog prototype. By finding the coefficients of a first order system, an information can be retrieved to solve the second order system. A general form solution is obtained, where only elementary functions are used and where the cutoff frequency can tend to infinity.

1. INTRODUCTION

When mapping from the analog domain to the digital domain, traditional methods like the bilinear transform are very popular due to their ease of directly replacing the variables. Unfortunately, they suffer from bad matching at high cutoff frequency and from cramping at Nyquist. Other methods try to solve the magnitude matching and the cramping, but often involve system dependant computation methods, complicated arithmetics and neglet the phase response. The goal of this paper is to come up with a simple and accurate solution for any given system, and simple solutions always come from a simple development. In Section 2.1, by knowing the location of the zeros of a discrete high pass filter, we can easily solve its first order form. In Section 3.1, we then do the same for its second order form while using a precious information appearing in the first order solution. For finding the coefficients, only systems of polynomial equations of degree not higher than two were used, resulting in elegant formulæ. Unlike other methods, the cutoff frequency is not bounded by the sampling frequency.

2. FIRST ORDER FILTER

2.1. HIGH PASS

A first order analog high pass filter, with cutoff frequency f_c, has a magnitude response

$$\Gamma(x) = \frac{1}{\sqrt{\left(\frac{f_c}{x}\right)^2 + 1}} \tag{2.1}$$

A digital bilinear filter, with sampling frequency f_s, has a discrete transfer function

$$H(z) = \frac{a_0 + a_1 z^{-1}}{b_0 + b_1 z^{-1}} = A\frac{1 + \alpha z^{-1}}{1 + \beta z^{-1}} \tag{2.2}$$

with a magnitude response

$$\Omega(x) = A\sqrt{\frac{\alpha^2 + 2\alpha\cos\left(\frac{2\pi x}{f_s}\right) + 1}{\beta^2 + 2\beta\cos\left(\frac{2\pi x}{f_s}\right) + 1}} \tag{2.3}$$

We introduce $\omega \coloneqq f_s/f_c$. The first two conditions to match the analog prototype are:

1. same gain at $x = 0$,
2. same slope at $x = 0$.

$$\left\| \begin{aligned} \tfrac{\mathrm{d}^0}{\mathrm{d}x^0}\Omega(x=0) &\overset{!}{=} \tfrac{\mathrm{d}^0}{\mathrm{d}x^0}\Gamma(x=0) \\ \tfrac{\mathrm{d}^1}{\mathrm{d}x^1}\Omega(x=0) &\overset{!}{=} \tfrac{\mathrm{d}^1}{\mathrm{d}x^1}\Gamma(x=0) \end{aligned} \right. \implies \left\| \begin{aligned} \alpha &= -1 \\ A &= (1+\beta)\tfrac{\omega}{2\pi} \end{aligned} \right. \tag{2.4}$$

To find β, we want a gain match at f_s/σ, where σ is a constant.

$$\begin{gathered} \Omega\left(\frac{f_s}{2}\right) \overset{!}{=} \Gamma\left(\frac{f_s}{\sigma}\right) \\ \implies \quad \beta = \frac{\pi - \nu}{\pi + \nu}, \qquad \nu \coloneqq \sqrt{\omega^2 + \sigma^2} \end{gathered} \tag{2.5}$$

2.2. FURTHER FILTER TYPES

Let

$$\varphi(x) = \frac{\pi - \sqrt{x^2 + \sigma^2}}{\pi + \sqrt{x^2 + \sigma^2}} \tag{2.6}$$

with

$$\begin{aligned}\lim_{x\to 0}\varphi(x) &= \frac{\pi-\sigma}{\pi+\sigma}\\ \lim_{x\to\infty}\varphi(x) &= -1\end{aligned}\tag{2.7}$$

Different first order filters can be created in the form

$$H(z) = G\cdot\left(1+\beta\right)\left(\frac{1+\alpha z^{-1}}{1+\beta z^{-1}}\right)\tag{2.8}$$

with the following coefficients:

Table 2.1
First order coefficients for various systems.

	α	β	G
High Pass	$\varphi(\infty)$	$\varphi(\omega)$	$\frac{\omega}{2\pi}$
Low Pass	$\varphi(0)$	$\varphi(\omega)$	$\frac{1}{1+\alpha}$
High Shelf	$\varphi\left(\omega g^{1/2}\right)$	$\varphi\left(\omega g^{-1/2}\right)$	$\frac{1}{1+\alpha}$
Low Shelf	$\varphi\left(\omega g^{-1/2}\right)$	$\varphi\left(\omega g^{1/2}\right)$	$\frac{g}{1+\alpha}$
All Pass	$\varphi(\omega)^{-1}$	$\varphi(\omega)$	$\frac{1}{1+\alpha}$

The frequency mapping of $H(z)$ is

$$\psi(\omega) = \frac{f_s}{2\pi}\arccos\left(\frac{\omega^2-\sigma^2-\pi^2}{\omega^2-\sigma^2+\pi^2}\right)\tag{2.9}$$

and its inverse mapping is

$$\psi^{-1}(\omega) = \frac{f_s}{\sqrt{\pi^2\cot^2\left(\frac{\pi}{\omega}\right)+\sigma^2}}\tag{2.10}$$

As shown in **Figure 2.1**, the best match of the magnitude response is achieved when $\sigma\in\left[2,\sqrt{2/3}\pi\right]$. This is the interval where $\psi(\omega)$ intersects f_c. The best match of the phase response is achieved when $\sigma = 0$.

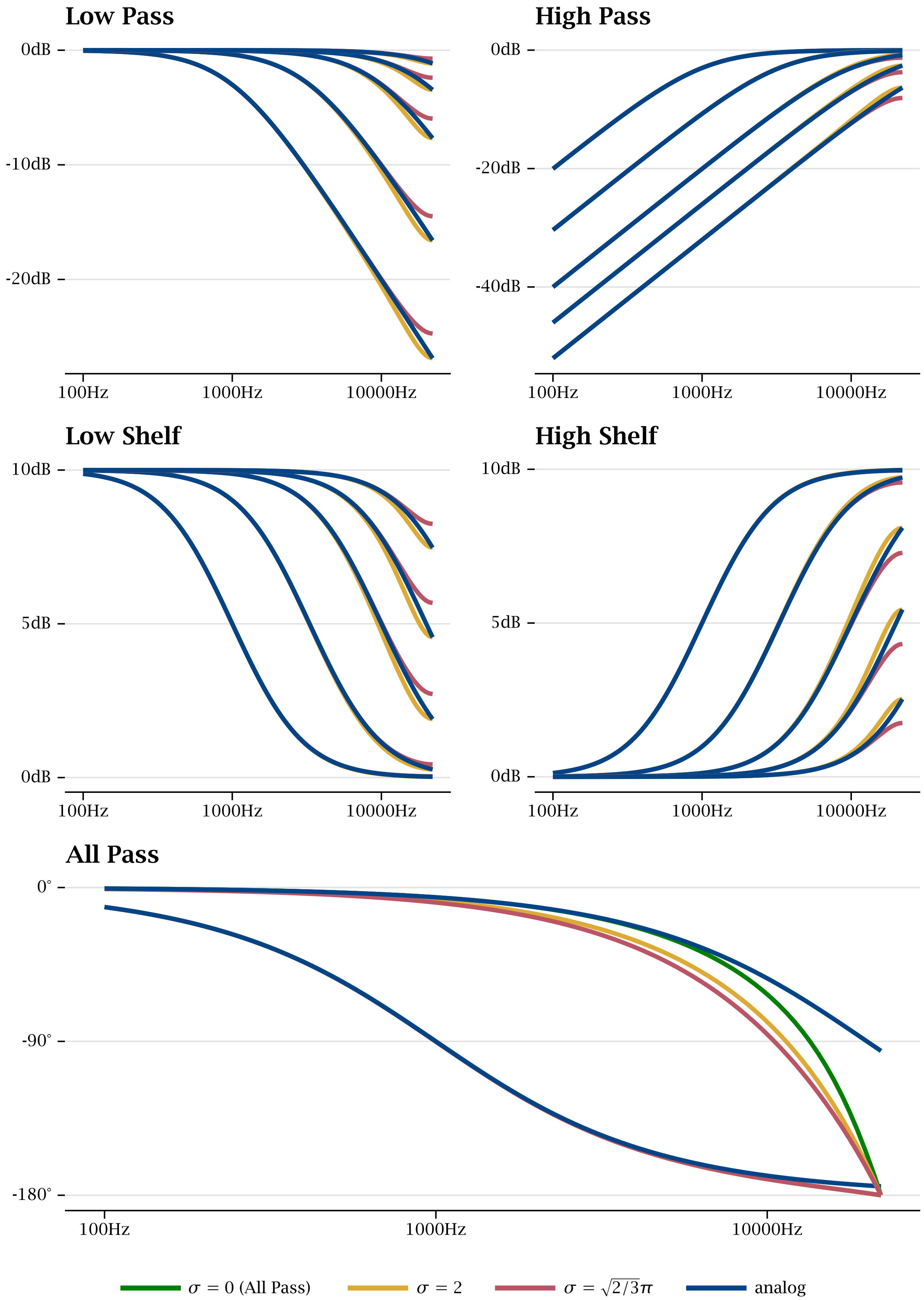

Figure 2.1. Different first order filters with $f_c = [\underline{1000}; 3300; 10000; \underline{20000}; 40000]$Hz, $f_s = 44100$Hz, $\quad g = 10$dB.

3. SECOND ORDER FILTER

3.1. HIGH PASS

A second order analog high pass filter, with damping ratio ζ, has a magnitude response

$$\Gamma(x) = \frac{1}{\sqrt{\left(\frac{f_c}{x}\right)^4 + 2\left(\frac{f_c}{x}\right)^2\left(2\zeta^2 - 1\right) + 1}} \tag{3.1}$$

A digital biquadratic filter has a discrete transfer function

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{b_0 + b_1 z^{-1} + b_2 z^{-2}} = A\frac{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2}} \tag{3.2}$$

with a magnitude response

$$\Omega(x) = A\sqrt{\frac{4\alpha_2 \cos^2\left(\frac{2\pi x}{f_s}\right) + 2\alpha_1(\alpha_2 + 1)\cos\left(\frac{2\pi x}{f_s}\right) + \alpha_1^2 + (\alpha_2 - 1)^2}{4\beta_2 \cos^2\left(\frac{2\pi x}{f_s}\right) + 2\beta_1(\beta_2 + 1)\cos\left(\frac{2\pi x}{f_s}\right) + \beta_1^2 + (\beta_2 - 1)^2}} \tag{3.3}$$

We want the gain, slope and concavity to match the analog prototype at $x = 0$, giving us the system of equation

$$\left\Vert\begin{array}{l} \frac{\mathrm{d}^0}{\mathrm{d}x^0}\Omega(x = 0) \overset{!}{=} \frac{\mathrm{d}^0}{\mathrm{d}x^0}\Gamma(x = 0) \\ \frac{\mathrm{d}^1}{\mathrm{d}x^1}\Omega(x = 0) \overset{!}{=} \frac{\mathrm{d}^1}{\mathrm{d}x^1}\Gamma(x = 0) \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Omega(x = 0) \overset{!}{=} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Gamma(x = 0) \end{array}\right. \implies \left\Vert\begin{array}{ll} \alpha_1 & = -2 \\ \alpha_2 & = 1 \\ A & = \left(1 + \beta_1 + \beta_2\right)\left(\frac{\omega}{2\pi}\right)^2 \end{array}\right. \tag{3.4}$$

To find β_1, we use the same condition as in (2.5).

$$\begin{gathered} \Omega\left(\frac{f_s}{2}\right) \overset{!}{=} \Gamma\left(\frac{f_s}{\sigma}\right) \\ \implies \quad \beta_1 = \frac{\pi^2 - \nu}{\pi^2 + \nu}(\beta_2 + 1), \qquad \nu \coloneqq \sqrt{\omega^4 + 2\sigma^2\omega^2(2\zeta^2 - 1) + \sigma^4} \end{gathered} \tag{3.5}$$

To solve the last unknown β_2, we'll use the frequency mapping function ψ from (2.9). We want the magnitude response at the frequency $\psi(\omega)$ to be equal to the resonance $Q \coloneqq 1/2\zeta$.

$$\begin{gathered} (\Omega \circ \psi)(\omega) \overset{!}{=} \frac{1}{2\zeta} \\ \implies \quad \beta_2 = \frac{\pi^2 + \nu - \pi\sqrt{2}\sqrt{\nu + \kappa}}{\pi^2 + \nu + \pi\sqrt{2}\sqrt{\nu + \kappa}}, \qquad \kappa \coloneqq \omega^2\left(2\zeta^2 - 1\right) + \sigma^2 \end{gathered} \tag{3.6}$$

3.2. FURTHER FILTER TYPES

Let

$$
\begin{aligned}
\nu(x,y) &= \sqrt{x^4 + 2\sigma^2 x^2 (2y^2 - 1) + \sigma^4} \\
\kappa(x,y) &= x^2\left(2y^2 - 1\right) + \sigma^2 \\
\varphi_1(x,y) &= \frac{2\pi^2 - 2\nu(x,y)}{\pi^2 + \nu(x,y) + \pi\sqrt{2}\sqrt{\nu(x,y) + \kappa(x,y)}} \\
\varphi_2(x,y) &= \frac{\pi^2 + \nu(x,y) - \pi\sqrt{2}\sqrt{\nu(x,y) + \kappa(x,y)}}{\pi^2 + \nu(x,y) + \pi\sqrt{2}\sqrt{\nu(x,y) + \kappa(x,y)}}
\end{aligned}
\tag{3.7}
$$

Different second order filters can be created in the form

$$
H(z) = G \cdot \left(1 + \beta_1 + \beta_2\right)\left(\frac{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}\right) \tag{3.8}
$$

with the following coefficients:

Table 3.1
Second order coefficients for various systems; with φ from Section 2.2.

	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{G}$
High Pass	$\varphi(\infty) + \varphi(\infty)$	$\varphi(\infty) \cdot \varphi(\infty)$	$\varphi_1(\omega,\zeta)$	$\varphi_2(\omega,\zeta)$	$\frac{\omega^2}{4\pi^2}$
Band Pass	$\varphi(0) + \varphi(\infty)$	$\varphi(0) \cdot \varphi(\infty)$	$\varphi_1(\omega,\zeta)$	$\varphi_2(\omega,\zeta)$	$\frac{\omega}{2\pi}\frac{2\zeta}{1+\varphi(0)}$
Low Pass	$\varphi(0) + \varphi(0)$	$\varphi(0) \cdot \varphi(0)$	$\varphi_1(\omega,\zeta)$	$\varphi_2(\omega,\zeta)$	$\frac{1}{1+\alpha_1+\alpha_2}$
Band Stop	$\varphi_1(\omega,0)$	$\varphi_2(\omega,0)$	$\varphi_1(\omega,\zeta)$	$\varphi_2(\omega,\zeta)$	$\frac{1}{1+\alpha_1+\alpha_2}$
High Shelf	$\varphi_1\left(\omega g^{1/4},\zeta\right)$	$\varphi_2\left(\omega g^{1/4},\zeta\right)$	$\varphi_1\left(\omega g^{-1/4},\zeta\right)$	$\varphi_2\left(\omega g^{-1/4},\zeta\right)$	$\frac{1}{1+\alpha_1+\alpha_2}$
Low Shelf	$\varphi_1\left(\omega g^{-1/4},\zeta\right)$	$\varphi_2\left(\omega g^{-1/4},\zeta\right)$	$\varphi_1\left(\omega g^{1/4},\zeta\right)$	$\varphi_2\left(\omega g^{1/4},\zeta\right)$	$\frac{g}{1+\alpha_1+\alpha_2}$
Peaking	$\varphi_1\left(\omega,\zeta g^{1/2}\right)$	$\varphi_2\left(\omega,\zeta g^{1/2}\right)$	$\varphi_1\left(\omega,\zeta g^{-1/2}\right)$	$\varphi_2\left(\omega,\zeta g^{-1/2}\right)$	$\frac{1}{1+\alpha_1+\alpha_2}$
All Pass	$\frac{\varphi_1(\omega,\zeta)}{\varphi_2(\omega,\zeta)}$	$\varphi_2(\omega,\zeta)^{-1}$	$\varphi_1(\omega,\zeta)$	$\varphi_2(\omega,\zeta)$	$\frac{1}{1+\alpha_1+\alpha_2}$

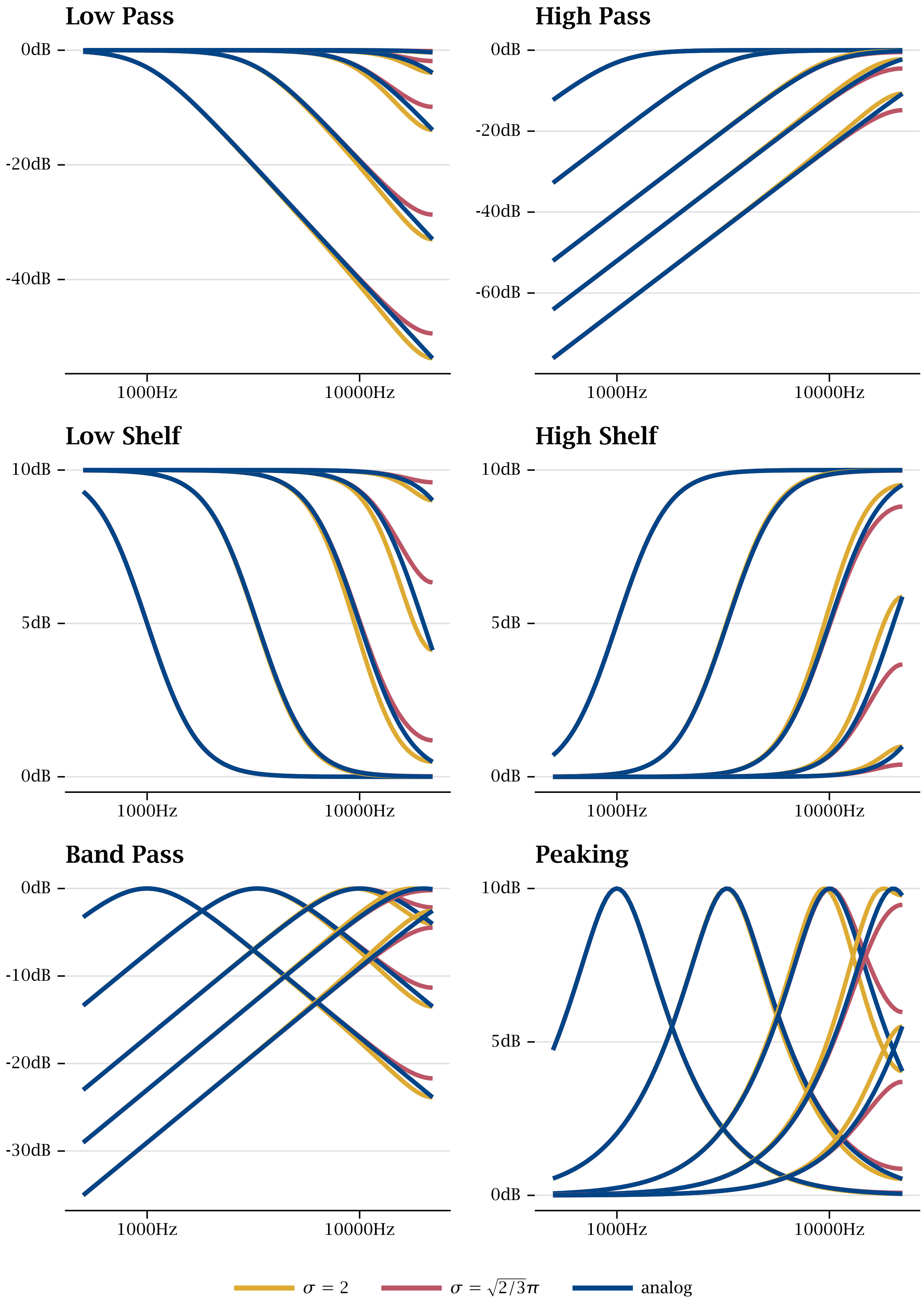

Figure 3.1. Different second order filters with $f_c = [1000; 3300; 10000; 20000; 40000]$Hz, $f_s = 44100$Hz, $g = 10$dB, $\zeta = \sqrt{2}/2$.

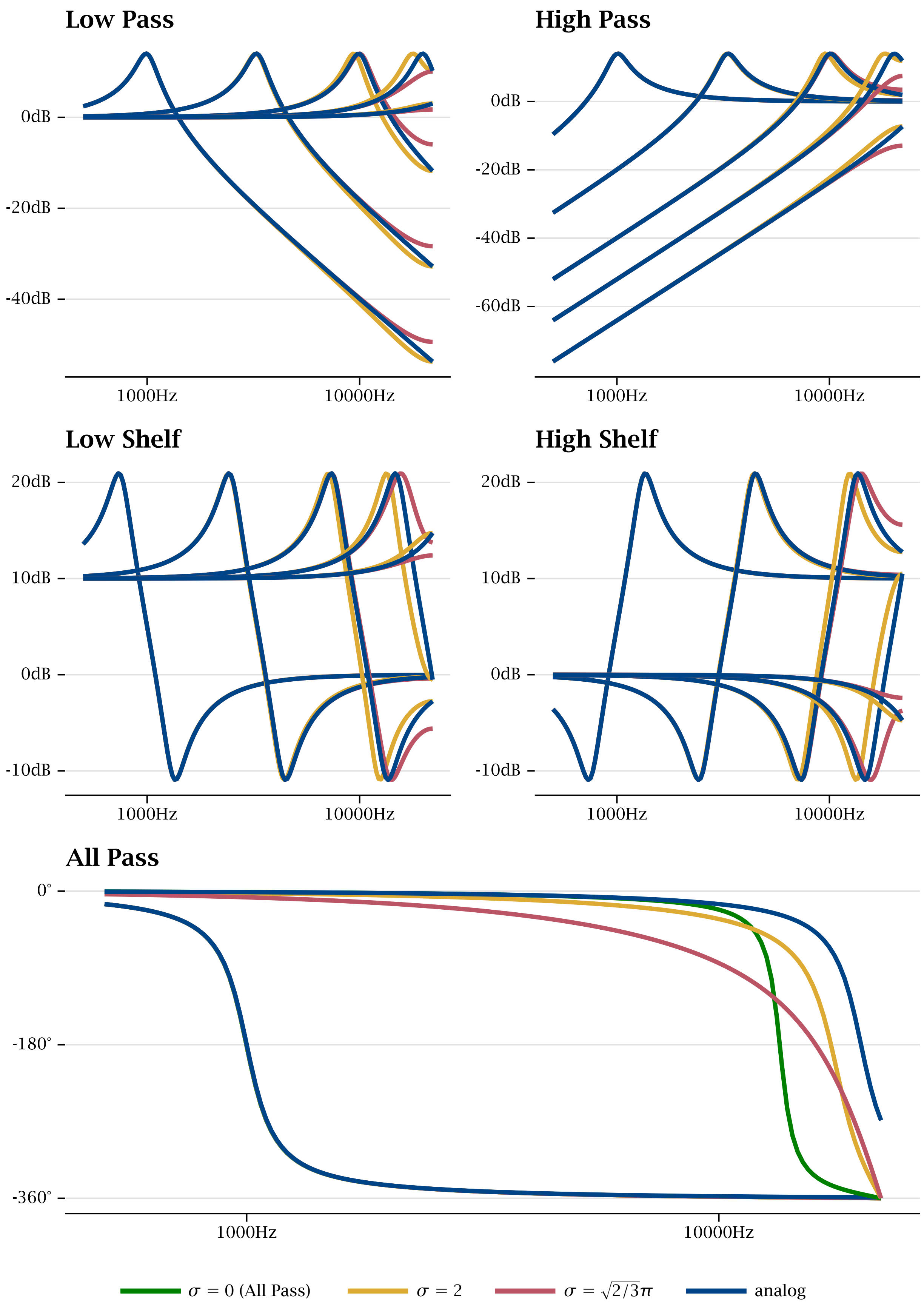

Figure 3.2. Different second order filters with $f_c = [\underline{1000}; 3300; 10000; \underline{20000}; 40000]$Hz, $f_s = 44100$Hz, $\quad g = 10$dB, $\quad \zeta = 1/10$.

4. HIGHER ORDER FILTER

4.1. GENERAL FORM

From the fundamental theorem of algebra, it is known that every real polynomial of order N can be decomposed into linear and quadratic real factors.

$$\sum_{n=0}^{N} a_n s^n = \underbrace{(a_{0,0} + a_{0,1} s)}_{\text{if } N \text{ odd}} \prod_{n=1}^{\lfloor N/2 \rfloor} \sum_{m=0}^{2} a_{n,m} s^m, \qquad a_n, a_{n,m} \in \mathbb{R} \tag{4.1}$$

A first order s-plane polynomial can be mapped to the z-plane with

$$\begin{aligned} as + 1 &\multimap\bullet \frac{1 + cz^{-1}}{1 + c} \\ s &\multimap\bullet \left(1 - z^{-1}\right)\frac{\omega}{2\pi} \end{aligned} \tag{4.2}$$

where

$$c = \begin{cases} \varphi(\omega a) & a \geq 0, \\ \varphi(\omega a)^{-1} & a \leq 0. \end{cases} \tag{4.3}$$

A second order s-plane polynomial, with real roots, is a product of two linear factors and the mapping from (4.2) can then be applied. Otherwise, for complex roots, we use the mapping

$$as^2 + bs + 1 \multimap\bullet \frac{1 + c_1 z^{-1} + c_2 z^{-2}}{1 + c_1 + c_2} \tag{4.4}$$

where

$$c_{\{1,2\}} = \begin{cases} \left\{\varphi_1(x, y) \quad , \varphi_2(x, y) \quad \right\} & b \geq 0, \\ \left\{\varphi_1(x, y)\varphi_2(x, y)^{-1}, \varphi_2(x, y)^{-1}\right\} & b \leq 0. \end{cases} \tag{4.5}$$

and

$$x = \omega\sqrt{a}, \qquad y = \frac{b}{2\sqrt{a}} \tag{4.6}$$

For the degenerative cases of when a or a and b equal to 0, we assume

$$\begin{aligned} bs + 1 &= (as + 1)(bs + 1)\Big|_{a=0} \\ 1 &= (as + 1)(as + 1)\Big|_{a=0} \end{aligned} \tag{4.7}$$

4.2. LADDER EXAMPLE

A fourth order low pass ladder filter with feedback coefficient K has the continuous transfer function

$$H(s) = \frac{1 + K^4}{(1+s)^4 + K^4} \tag{4.8}$$

Then, with

$$a = 0, \qquad a_{\{1,2\}} = \frac{1}{\sqrt{K^2 \pm \sqrt{2}K + 1}}, \qquad b_{\{1,2\}} = 2 \pm \sqrt{2}K \tag{4.9}$$

it can be factorized and mapped as follow:

$$\begin{array}{rccc}
H(s) = & \Big(as + 1\Big)^4 & \times & \left(\prod_{n=1}^{2} \Big(a_n^2 s^2 + a_n^2 b_n s + 1\Big)\right)^{-1} \\
& \big\downarrow & & \big\downarrow \\
H(z) = & \left(\dfrac{1 + \varphi(\omega a) z^{-1}}{1 + \varphi(\omega a)}\right)^4 & \times & \left(\prod_{n=1}^{2} \dfrac{1 + \sum_{m=1}^{2} \varphi_m\Big(\omega a_n, \frac{1}{2} a_n b_n\Big) z^{-m}}{1 + \sum_{m=1}^{2} \varphi_m\Big(\omega a_n, \frac{1}{2} a_n b_n\Big)}\right)^{-1}
\end{array} \tag{4.10}$$

5. FINAL REMARKS

- For elementary systems like the low-, high-, and band-pass filter, i.e. when the zeros of the continuous transfer function are equal to 0 or infinity, the zeros of the discrete version are just constants.
- Freedom of optimization and matching preference is given via the σ constant. Setting it to 0 improves a lot the computation of the coefficients, but at the cost of cramping at Nyquist. For small f_c, that drawback is negligible.
- Frequency compensation from (2.10) can be used to obtain the same cutoff point as in the analog system and to improve the matching over the whole frequency spectrum.
- If the roots of a continuous system are on the right half of the s-plane, then at a very specific cutoff frequency, φ and φ_2 can be equal to 0, implying a division by 0 in (4.3) and (4.5).
- We notice in (3.7) that φ_1 and φ_2 are closely related to each other, sharing the same denominator d. Interestingly enough, $1 + \varphi_1 + \varphi_2$ can also be written as $4\pi^2/d$, which intuitively thinking, isn't a coincidence and could be exploited.