ON THE STATE SPACE OF A LINEAR DIGITAL FILTER

Yuriy Ivantsov
Sofia, Bulgaria
y@ivantsovy.com
08 October 2025

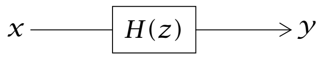
Abstract. The time variant behaviour of a linear discrete system is studied, using the state space representation. By observing the flow of energy inside the capacitor of an analog filter, a first and second order system are brought into optimised time variant and time invariant recursive stable structures. The transformation of a continuous state space into its discrete form is then presented, with the eigenspaces playing the key role.

1. INTRODUCTION

Once the transfer function of a linear digital system is obtained, it is of interest, for realtime simulations, to make use of it. For recursive filters, there exists, however, an infinite amount of different structures. Some of them are simpler to implement, faster to execute, or more robust regarding numerical stability. Therefore, when solving problems, it is important to start with its simplest version possible; gather all the information, solve, and continue with its higher order form. In Section 3, the recursive structure is brought into the state space form and a direct relationship with the analog RC circuit is found. Stability is then studied in Section 4, focused more particularly on time variant systems. From there, we discretise a continuous state space that is known to be stable. Moreover, we methodically expand this derived form into something modular, without altering its frequency response. Finally, thanks to the coefficients in [1], all the structures presented here can be used by any filter type, with very fast execution.

2. DIRECT FORM

A linear time invariant system can be characterised by its transfer function $H(z)$, the latter being the direct relation between the input x and the output y.



(2.1)

This means, given a vector x, we can predict the vector y simply by looking at $H(z)$, or, more precisely, at its poles and zeros. We refer to this property as determinacy; one of the mainstay of ordinary differential equations. However, for realtime causal systems, the input vector comes sample by sample. Therefore, in order to bring a discrete transfer function into a such configuration, there exists a simple and intuitive transformation, as shown in Table 2.1, using the low pass structure from [1][1]:

Table 2.1.
Direct Form I mapping of a first and second order transfer function.

	First order	Second order
$H(z)$	$\frac{1+\beta}{1+\alpha} \cdot \frac{1+\alpha z^{-1}}{1+\beta z^{-1}}$	$\frac{1+\beta_1+\beta_2}{1+\alpha_1+\alpha_2} \cdot \frac{1+\alpha_1 z^{-1}+\alpha_2 z^{-2}}{1+\beta_1 z^{-1}+\beta_2 z^{-2}}$
y_n	$\frac{1+\beta}{1+\alpha} \cdot (\mathrm{x}_n + \alpha \mathrm{x}_{n-1}) - \beta \mathrm{y}_{n-1}$	$\frac{1+\beta_1+\beta_2}{1+\alpha_1+\alpha_2} \cdot (\mathrm{x}_n + \alpha_1 \mathrm{x}_{n-1} + \alpha_2 \mathrm{x}_{n-2}) - \beta_1 \mathrm{y}_{n-1} - \beta_2 \mathrm{y}_{n-2}$

The pattern is apparent. Each z^{-n} acts as a state from the past, also known as delay element. The literature calls this recursive sequence Direct Form I. Although being a mathematically correct model in an ideal world, it holds some weaknesses in the real world. These are, for a system of order m:

1. Optimum number of states:
   - $2m$ states.
2. Optimum number of elementary arithmetic operators:
   - $2m$ additions.
   - $2m + 1$ multiplications.
3. Numerical instability for low cut systems with a very small cutoff frequency.
4. Instability for $m > 1$ for time varying coefficients.

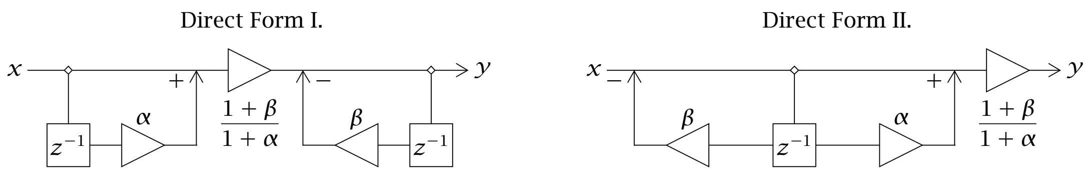


Figure 2.1. Structures directly derived from the transfer function $H(z)$.

With a simple algebraic manipulation, we can halve the number of states and turn the Direct Form I into the Direct Form II, as shown in Figure 2.1. We have solved point 1, but aggravated point 3. Therefore, simple and random attempts are not the keys to solve the problem.

[1]This paper will mostly use the coefficients from [1], but the reasoning will not be limited to these. Any variable not defined here is defined in [1].

3. STATE SPACE

3.1. DEFINITION

A linear system of order m in the state space representation has the general form

$$\begin{aligned} \mathrm{y}_n &= \mathrm{A}_n \mathrm{x}_n + \mathrm{B}_n \mathrm{z}_n \\ \mathrm{z}_{n+1} &= \mathrm{C}_n \mathrm{x}_n + \mathrm{D}_n \mathrm{z}_n \end{aligned} \tag{3.1}$$

The m dimensional real vector z_n is composed of a minimum set of states that uniquely describe a system, given its state matrices. These are $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}_n$, containing the coefficients of the system with

$$\mathrm{A}_n \in \mathbb{R}^{1\times 1}, \qquad \mathrm{B}_n \in \mathbb{R}^{1\times m}, \qquad \mathrm{C}_n \in \mathbb{R}^{m\times 1}, \qquad \mathrm{D}_n \in \mathbb{R}^{m\times m} \tag{3.2}$$

The transfer function is

$$H_m(z) = \mathrm{A} + \mathrm{B} \cdot (z - \mathrm{D})^{-1}\mathrm{C} \tag{3.3}$$

Alternatively, for a first and second order system, with $\operatorname{tr} \cdot$ and $\det \cdot$ denoting the trace and determinant of a matrix, respectively, we get the convenient form

$$\begin{aligned} H_1(z) &= \mathrm{A} \cdot \frac{1 + \left(\mathrm{BCA}^{-1} - \operatorname{tr}\mathrm{D}\right) z^{-1}}{1 - \operatorname{tr}\mathrm{D}\, z^{-1}} \\ H_2(z) &= \mathrm{A} \cdot \frac{1 + \left(\mathrm{BCA}^{-1} - \operatorname{tr}\mathrm{D}\right) z^{-1} + \left(\mathrm{B} \cdot (\mathrm{I} - \mathrm{D})^{-1}\mathrm{C} - \mathrm{BCA}^{-1} + \det\mathrm{D}\right) z^{-2}}{1 - \operatorname{tr}\mathrm{D}\, z^{-1} + \det\mathrm{D}\, z^{-2}} \end{aligned} \tag{3.4}$$

If we ignore the pole independent scalar $G^{-1} = 1 + \alpha$ from our first order transfer function and solve the equation in the state space, we get

$$H_1(z) = (1 + \beta) \cdot \frac{1 + \alpha z^{-1}}{1 + \beta z^{-1}} \quad \Longrightarrow \quad \begin{cases} \mathrm{A} = 1 + \beta \\ \mathrm{D} = -\beta \\ \mathrm{BC} = (1 + \beta)(\alpha - \beta) \end{cases} \tag{3.5}$$

The set of solutions is infinite. Since, unlike the output matrix B, the input matrix C is directly related to the evolution of the states, it is important to understand its meaning. Apart from having a useful property for controllability in control theory, it is actually directly linked to the passive elements of an electronic circuit.

3.2. ANALOGUE ANALOGY

A first order analogue filter is made of a resistance R and a capacitance C. The transfer function $H(z)$ depends of these two elements, which can be combined into the complex impedance $Z = R + i/C$.

With a cutoff period of $2\pi RC$, it is of interest, for time variant systems, to know which one between the real and imaginary part of Z is varying. Indeed, based on this dilemma, the circuit shall react differently.

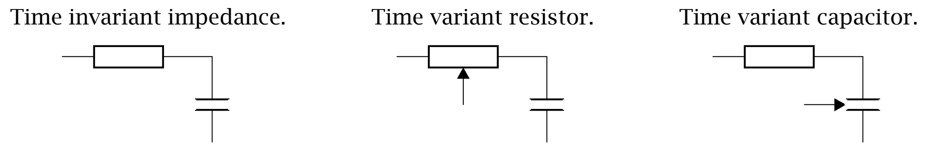


Figure 3.1. Analogue filter with different time behaviours.

Physically speaking, the state z_n can be compared to the electric charges Q_t stored on the conductors of a capacitor. These charges converge to the product of the capacitance C and the voltage U between the conductors. By assigning U to the input x, we can emulate the charging in the state space.

$$\begin{aligned} Q_t &\xrightarrow{t\to\infty} U \cdot C \\ \mathrm{z}_n &\xrightarrow{n\to\infty} U \cdot (\mathrm{I}-\mathrm{D})^{-1}\mathrm{C} \end{aligned} \tag{3.6}$$

We see that a constant imaginary impedance implies a constant matrix $(\mathrm{I}-\mathrm{D})^{-1}\mathrm{C}$. On the other hand, a constant real impedance implies a constant matrix C. If we normalise these constants to 1 and define $\mathrm{Q} \coloneqq (\mathrm{I}-\mathrm{D})^{-1}\mathrm{C}$ as the charge matrix, the state space can be completed for both time variant cases.

$$\begin{aligned} \Im\mathfrak{m}\{Z\} = 1 \quad &\Longrightarrow \quad \mathrm{Q} = 1 \quad \Longrightarrow \quad \begin{cases} \mathrm{B} = \alpha - \beta \\ \mathrm{C} = 1 + \beta \end{cases} \\ \Re\mathfrak{e}\{Z\} = 1 \quad &\Longrightarrow \quad \mathrm{C} = 1 \quad \Longrightarrow \quad \begin{cases} \mathrm{B} = (\alpha - \beta)(1+\beta) \\ \mathrm{C} = 1 \end{cases} \end{aligned} \tag{3.7}$$

It appears that the Direct Form II in Figure 2.1 also simulates a circuit with a time varying imaginary impedance. Whether such behaviour is desired is an artistic or engineering preference. Nevertheless, instead of evaluating the state z_n and output y_n according to (3.1), it is possible to simplify the process with

$$\vartheta_{\Im\mathfrak{m}} = (1+\beta)(\mathrm{x}_n - \mathrm{z}_n), \qquad \vartheta_{\Re\mathfrak{e}} = \mathrm{x}_n - (1+\beta)\cdot \mathrm{z}_n \tag{3.8}$$

as shown in the following tables:

Table 3.1.
State space update with $\Im\mathfrak{m}\{Z\} = 1$.

	y_n	z_{n+1}
High Pass	$G \cdot \vartheta_{\Im\mathfrak{m}}$	$\mathrm{z}_n + \vartheta_{\Im\mathfrak{m}}$
Other	$G \cdot \vartheta_{\Im\mathfrak{m}} + \mathrm{z}_n$	$\mathrm{z}_n + \vartheta_{\Im\mathfrak{m}}$

Table 3.2.
State space update with $\Re\mathfrak{e}\{Z\} = 1$.

	y_n	z_{n+1}
High Pass	$(1+\beta)(G \cdot \vartheta_{\Re\mathfrak{e}})$	$\mathrm{z}_n + \vartheta_{\Re\mathfrak{e}}$
Other	$(1+\beta)(G \cdot \vartheta_{\Re\mathfrak{e}} + \mathrm{z}_n)$	$\mathrm{z}_n + \vartheta_{\Re\mathfrak{e}}$

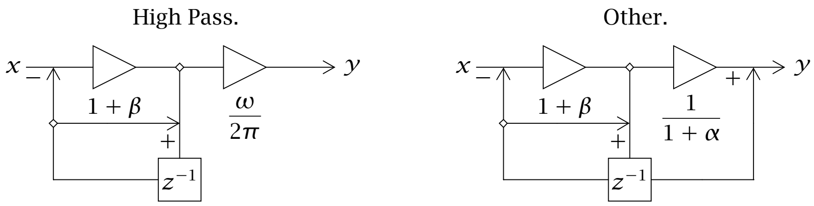


Figure 3.2. Optimised state space update with $\mathfrak{Im}\{Z\} = 1$.

With a constant imaginary impedance, comparing against the Direct Form I, we have created a structure where we have minimised the number of states, bounded the norm of the state vector z_n below the norm of the input vector x_n and traded one multiplication for one addition.

3.3. SECOND ORDER

Solving for the second order system, we get

$$H_2(z) = (1 + \beta_1 + \beta_2) \cdot \frac{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}$$
$$\implies \begin{cases} \mathrm{A} = 1 + \beta_1 + \beta_2 \\ \operatorname{tr}\mathrm{D} = -\beta_1 \\ \det\mathrm{D} = \beta_2 \end{cases} \implies \begin{cases} \mathrm{BC} = (1 + \beta_1 + \beta_2)(\alpha_1 - \beta_1) \\ \mathrm{BQ} = \alpha_1 + \alpha_2 - \beta_1 - \beta_2 \end{cases} \tag{3.9}$$

We see that, where, in the first order case, we had only one degree of freedom, here, we have four of them. Nonetheless, we will attempt to find a solution by emphasising simplicity. Let us restrict the entries of the state matrices over the additive group $(\{1, \alpha_1, \alpha_2, \beta_1, \beta_2\}, +)$, allowing only additions and subtractions as arithmetic operators. This implies, for the charge matrix Q, to be restricted over the set $\{0, 1\}$, excluding the null matrix. With an order $m = 2$, we have $2^m - 1 = 3$ unique permutations:

$$\mathrm{Q}_1 = \left\|\begin{matrix}1\\0\end{matrix}\right\|, \quad \mathrm{Q}_2 = \left\|\begin{matrix}0\\1\end{matrix}\right\|, \quad \mathrm{Q}_3 = \left\|\begin{matrix}1\\1\end{matrix}\right\| \tag{3.10}$$

For the input matrix C, a restriction over the set $\{0, 1 + \beta_1 + \beta_2\}$ is implied, giving us the combination pairs

$$\begin{aligned} \mathrm{C}_1 &= \left\|\begin{matrix}0\\1 + \beta_1 + \beta_2\end{matrix}\right\| \equiv \mathrm{Q}_1 \\ \mathrm{C}_2 &= \left\|\begin{matrix}1 + \beta_1 + \beta_2\\1 + \beta_1 + \beta_2\end{matrix}\right\| \equiv \mathrm{Q}_2 \\ \mathrm{C}_3 &= \left\|\begin{matrix}1 + \beta_1 + \beta_2\\0\end{matrix}\right\| \equiv \mathrm{Q}_3 \end{aligned} \tag{3.11}$$

After solving the state space for the different cases and ignoring all kind of symmetric forms, we get three complete systems, shown in Table 3.3.

Table 3.3.
Different second order state space systems.

	A	B	C	D
System 1	$\begin{Vmatrix}1+\beta_1+\beta_2\end{Vmatrix}$	$\begin{Vmatrix}\alpha_1+\alpha_2-\beta_1-\beta_2\\ \alpha_1-\beta_1\end{Vmatrix}^{\mathrm{T}}$	$\begin{Vmatrix}0\\ 1+\beta_1+\beta_2\end{Vmatrix}$	$\begin{Vmatrix}1 & 1\\ -1-\beta_1-\beta_2 & -1-\beta_1\end{Vmatrix}$
System 2	$\begin{Vmatrix}1+\beta_1+\beta_2\end{Vmatrix}$	$\begin{Vmatrix}\beta_2-\alpha_2\\ \alpha_1+\alpha_2-\beta_1-\beta_2\end{Vmatrix}^{\mathrm{T}}$	$\begin{Vmatrix}1+\beta_1+\beta_2\\ 1+\beta_1+\beta_2\end{Vmatrix}$	$\begin{Vmatrix}\beta_2 & -1-\beta_1-\beta_2\\ \beta_2 & -\beta_1-\beta_2\end{Vmatrix}$
System 3	$\begin{Vmatrix}1+\beta_1+\beta_2\end{Vmatrix}$	$\begin{Vmatrix}\alpha_1-\beta_1\\ \alpha_2-\beta_2\end{Vmatrix}^{\mathrm{T}}$	$\begin{Vmatrix}1+\beta_1+\beta_2\\ 0\end{Vmatrix}$	$\begin{Vmatrix}-\beta_1 & -\beta_2\\ 1 & 0\end{Vmatrix}$

Interestingly, System 3 shares the same eigenspaces as the second order Direct Form II; but instead of having constant real impedances, it has constant imaginary impedances. Furthermore, seemingly, System 2 is an extension of our optimised first order structure. Indeed, with

$$\vartheta = (1+\beta_1+\beta_2)(\mathrm{x}_n - \mathrm{z}_{1|n}) + \beta_2 \mathrm{z}_{2|n} \tag{3.12}$$

we can simplify the state space update as follow:

Table 3.4.
State space update for System 2.

	y_{n+1}	$\mathrm{z}_{1\|n+1}$	$\mathrm{z}_{2\|n+1}$
Low Cut	$G\cdot(\vartheta-\alpha_2\mathrm{z}_{2\|n})$	$\mathrm{z}_{1\|n}+\vartheta$	ϑ
Other	$G\cdot(\vartheta-\alpha_2\mathrm{z}_{2\|n})+\mathrm{z}_{1\|n}$	$\mathrm{z}_{1\|n}+\vartheta$	ϑ

Low cut will refer to any system with a discrete transfer function having at least one zero at $z=1$; i.e., a high pass or band pass filter. We clearly see the similarities between Figure 3.2 and Figure 3.3; they obey the natural beauteous symmetry and share the same number of arithmetic operators: $2m+1$ additions and $2m$ multiplications.

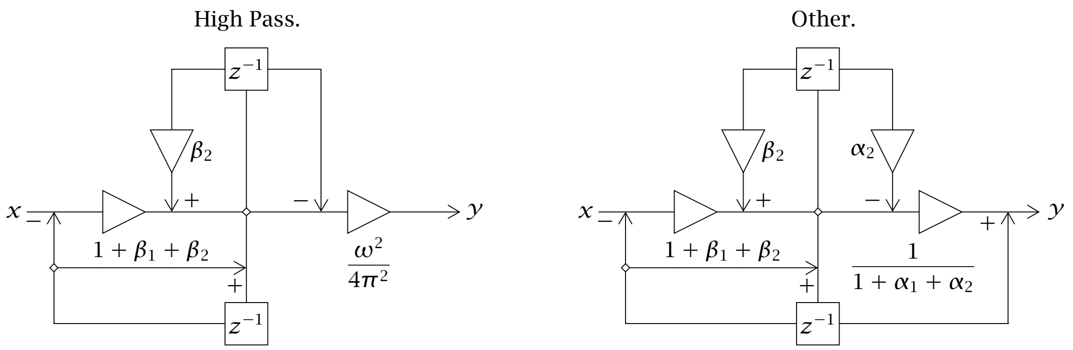


Figure 3.3. Optimised state space update for System 2.

Unfortunately, none of the systems from Table 3.3 are stable under modulation. Thus, some information is missing and more is to explore.

4. STABILITY

4.1. DEFINITIONS

Let $\|\cdot\|_p$ be the matrix norm induced by the vector norm $\|\cdot\|_p$ on $\mathbb{R}^m$ over $\mathbb{R}_+$.

Definition 1. A system is stable if a finite input x produces a finite output y.

$$\|\mathrm{x}_n\|_\infty < \infty \Longrightarrow \|\mathrm{y}_n\|_\infty < \infty, \qquad \forall n \in \mathbb{R}$$

Let S be a linear system defined as in (3.1) with $S \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{x}, \mathrm{y}, \mathrm{z}\}_n$ and $n \in \mathbb{N}$.

Lemma 1. The output vector y_n is finite if the matrices A_n and B_n, as well as the vector z_n, are finite.

Proof.

$$\|\mathrm{y}_n\|_\infty = \|\mathrm{A}_n\mathrm{x}_n + \mathrm{B}_n\mathrm{z}_n\|_\infty \leqslant \underbrace{\|\mathrm{A}_n\|_\infty}_{<\infty}\underbrace{\|\mathrm{x}_n\|_\infty}_{<\infty} + \underbrace{\|\mathrm{B}_n\|_\infty}_{<\infty}\underbrace{\|\mathrm{z}_n\|_\infty}_{<\infty}$$ □

The condition for stability is mostly about the state vector z_n. Since z_{n+1} is a recursive sequence, all we have to do is to show that it's defined everywhere and that it doesn't diverge.

Lemma 2. The sequence z_{n+1} is defined for all n if the matrices C_n and D_n are finite.

Proof.

$$\|\mathrm{z}_{n+1}\|_\infty = \|\mathrm{C}_n\mathrm{x}_n + \mathrm{D}_n\mathrm{z}_n\|_\infty \leqslant \underbrace{\|\mathrm{C}_n\|_\infty}_{<\infty}\underbrace{\|\mathrm{x}_n\|_\infty}_{<\infty} + \underbrace{\|\mathrm{D}_n\|_\infty}_{<\infty}\underbrace{\|\mathrm{z}_n\|_\infty}_{<\infty}$$ □

Lemma 3. The sequence z_{n+1} converges if the product of the matrices $\mathrm{D}_0, \mathrm{D}_1, \ldots, \mathrm{D}_{n-1}, \mathrm{D}_n$ converges to the null matrix.

Proof. Expending z_{n+1} and taking its limit, we get

$$\lim_{n\to\infty} \mathrm{z}_{n+1} = \lim_{n\to\infty}\left(\prod_{i=0}^{n} \mathrm{D}_i \cdot \mathrm{z}_0 + \sum_{i=0}^{n}\prod_{j=1}^{n-i} \mathrm{D}_j \cdot \mathrm{C}_i\mathrm{x}_i\right) = \prod_{i=1}^{\infty} \mathrm{D}_i \cdot \left(\mathrm{D}_0\mathrm{z}_0 + \sum_{i=0}^{\infty} \mathrm{C}_i\mathrm{x}_i\right)$$

which converges if there is absorption by 0. □

Hence, by implication of Lemmata 1-2-3, we get the following theorem:

Theorem 1. A system S is stable if the matrices A_n, B_n and C_n are finite and if the product of the matrices $\mathrm{D}_0, \mathrm{D}_1, \ldots, \mathrm{D}_{n-1}, \mathrm{D}_n$ converges to the null matrix.

Let us now assume that all the state matrices of the system S are finite. We can then give sufficient conditions on stability for the time variant and time invariant cases.

Proposition 1. A time invariant system S is stable if the spectral radius of the matrix D is less than 1.

Proof. If $\rho(\mathrm{D}) < 1$ and $n \to \infty$, then $\rho(\mathrm{D}^n) = \rho(\mathrm{D})^n \to 0$. It follows that D^n is nilpotent, thus $\mathrm{D}^n = 0$. □

For the time variant case, prudence is advised. Indeed, we can have a diverging product $D_0D_1 \cdots D_{n-1}$; but with $D_n = 0$, mathematically, the system is still stable. Yet, in the real world, the amount of energy accumulated inside a capacitor would break it. Or the state vector z_n could no longer be representable given some limited memory. Therefore, the best way to fight an enemy is to not have one.

Proposition 2. A time variant system S is stable if the spectral norm of the matrix D_n is less than 1.

Proof. Clearly, given any matrix norm $\|\cdot\|_p < 1$,

$$\prod \|D_n\|_p \implies \prod D_n \to 0$$

Since $\|D_n\|_2^2 = \rho\left(D_n D_n^T\right) \leqslant \|D_n D_n^T\|_p$, the lower bound to the matrix norm is given for $p = 2$. □

Nevertheless, stable systems with a spectral norm equal to 1 can fe found. This special case tells us that there exists a non null vector z, such that $\|D_n z\|_2 = \|z\|_2$; in other words, the transformation preserves the magnitude of the vector. The question is, however, if $\|D_n D_{n+1} \cdots z\|_2 < \|z\|_2$. We could, of course, sequentially compute the matrix product and see if the spectral norm is less than 1, but it would be much easier to prove it by exhaustion instead.

Example 1. Let there be given a real vector $z = \begin{Vmatrix} a \\ b \end{Vmatrix}$ with $a \neq 0$ and $b \neq 0$. Then, the only way to apply a transformation that preserves the magnitude is by rotation. A characteristic of such a matrix is a determinant equal to 1.

Example 2. Let us take the same vector z, but with $b = 0$ instead. We have a case where the 2-dimensional vector z lies in the span of the set $P^2 = \left\{\begin{Vmatrix} 1 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}\right\}$, but also in the span of the set $P^1 = \left\{\begin{Vmatrix} 1 \\ 0 \end{Vmatrix}\right\}$. If we prohibit rotation, then a magnitude preserving transformation on the vector z can be done by shifting. If the order of the shift gets reversed by a cycle, oscillation might happen. For instance, let us choose

$$M_1 = \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}, \qquad M_2 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}$$

Then the entries of the vector z can be shifted up and down in a cyclic manner:

$$M_1 z = \begin{Vmatrix} 0 \\ a \end{Vmatrix}, \quad M_2 M_1 z = \begin{Vmatrix} a \\ 0 \end{Vmatrix}, \quad M_1 M_2 M_1 z = \begin{Vmatrix} 0 \\ a \end{Vmatrix}, \quad \ldots$$

Instead of trying to identify a cyclic shift, it would be simpler to not allow it in the first place. In fact, that combinatorial phenomenon is not present in analog filters. A sufficient condition for it not to occur is to have all non diagonal entries of a matrix not equal to ± 1.

From exhaustion, the final proposition is obtained:

Proposition 3. The following holds:

$$\begin{cases} \|D_n\|_2 = 1 \\ \det(D_n) \neq 1 \quad \text{(no rotation)} \\ \max_{i,j} |D_{n|ij}| \neq 1 \quad \text{(no shift)} \end{cases} \implies \prod D_n \to 0$$

4.2. STATE VARIABLE FORM

There exists an infinite amount of stable time variant second order structures. Therefore, in order to start from somewhere, we shall take inspiration from simple continuous forms that are known to be stable. Indeed, such a system can be characterised by the following continuous transition matrix D_t and charge matrix Q:

$$D_t = \begin{Vmatrix} 0 & 1 \\ -1 & -2\zeta \end{Vmatrix} \cdot 2\pi f_c, \qquad Q = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \tag{4.1}$$

The literature calls this structure State Variable Form. We recall that the convergence of the time variant transition matrix is not only about its eigenvalues, but also about its eigenvectors. To discretise D_t, we simply eigen decompose it, transform its eigenvalues, and recompose back.

$$D_t = V_t \Lambda_t V_t^{-1} \implies D_n = V_t \mathrm{Tr}\{\Lambda_t\} V_t^{-1} = V_t \Lambda_n V_t^{-1} \tag{4.2}$$

In fact, with

$$V_t = \begin{Vmatrix} 1 & 1 \\ -\zeta + \sqrt{\zeta^2 - 1} & -\zeta - \sqrt{\zeta^2 - 1} \end{Vmatrix}, \qquad \Lambda_n = \frac{1}{2} \cdot \begin{Vmatrix} -\beta_1 + \sqrt{\beta_1^2 - 4\beta_2} & 0 \\ 0 & -\beta_1 - \sqrt{\beta_1^2 - 4\beta_2} \end{Vmatrix} \tag{4.3}$$

the discrete state space can be sequentially solved in the following way:

$$D_n = V_t \Lambda_n V_t^{-1} \implies C_n = (I - D_n)Q \implies B_n = \begin{Vmatrix} (1 + \beta_1 + \beta_2)(\alpha_1 - \beta_1) \\ \alpha_1 + \alpha_2 - \beta_1 - \beta_2 \end{Vmatrix}^T \cdot \begin{Vmatrix} C_n^T \\ Q^T \end{Vmatrix}^{-1} \tag{4.4}$$

Alternatively, we can substitute V_t in terms of elements of the set $\{1, \beta_1, \beta_2\}$. Depending on the eigenvalue mapping method, this might be tedious. Nevertheless, there is no requirement to exactly match V_t, as shown in Figure 5.1. Indeed, let us assume $\sigma = 0$. This gives us

$$\zeta = \frac{\sqrt{\nu + \kappa}}{\sqrt{2}\sqrt{\nu}} \tag{4.5}$$

Then, with $\mu := \sqrt{\nu}/\pi$, the following elegant state space can be obtained:

$$\begin{cases} A = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \\ B = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \cdot \begin{Vmatrix} -1 \\ -\mu \end{Vmatrix}^T + \begin{Vmatrix} 1 + \alpha_1 + \alpha_2 \\ 1 - \alpha_1 + \alpha_2 \end{Vmatrix}^T \cdot \begin{Vmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{Vmatrix} \\ C = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -\mu \end{Vmatrix} \cdot \frac{1}{2} \\ D = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -\mu \end{Vmatrix} \cdot \frac{1}{2} \cdot \begin{Vmatrix} -1 \\ -\mu \end{Vmatrix}^T + \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \end{cases} \tag{4.6}$$

Furthermore, with

$$\vartheta = (1 + \beta_1 + \beta_2)(x - \mathrm{z}_{1|n} - \mu \mathrm{z}_{2|n}) \tag{4.7}$$

the system can be updated as shown in Table 4.1:

Table 4.1.
Optimised state space update of the state variable form.

	y_n	$\mathrm{z}_{1\|n+1}$	$\mathrm{z}_{2\|n+1}$
Low Cut	$G \cdot \left(\vartheta + \frac{1-\alpha_1+\alpha_2}{\mu} \cdot \mathrm{z}_{2\|n}\right)$	$\mathrm{z}_{1\|n} + \frac{\vartheta}{2}$	$-\mathrm{z}_{2\|n} - \mu \cdot \frac{\vartheta}{2}$
Other	$G \cdot \left(\vartheta + \frac{1-\alpha_1+\alpha_2}{\mu} \cdot \mathrm{z}_{2\|n}\right) + \mathrm{z}_{1\|n}$	$\mathrm{z}_{1\|n} + \frac{\vartheta}{2}$	$-\mathrm{z}_{2\|n} - \mu \cdot \frac{\vartheta}{2}$

4.3. EIGENMORPH FORM

By neatly manipulating the eigenvector matrix V_t and the charge matrix Q, we can get, with $\eta \in \mathbb{R}$,

$$\mathrm{V}_t = \begin{Vmatrix} 1 & 1 \\ -\zeta + \sqrt{\zeta^2 - 1} - \eta & -\zeta - \sqrt{\zeta^2 - 1} - \eta \end{Vmatrix}, \qquad \mathrm{Q} = \begin{Vmatrix} 1 \\ -\eta \end{Vmatrix} \tag{4.8}$$

This form is then stable for time varying coefficients when $\eta \in \left[0, 2\frac{\sqrt{\nu+\kappa}}{\sqrt{2}\sqrt{\nu}}\right] \cong [0, 2\zeta]$. The interesting part, however, is that it nicely extends the state space of the state variable form in (4.6). At the expense of three more additions and one multiplication, we can have a modular structure that maintains its frequency response.

$$\begin{cases} \mathrm{A} = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \\ \mathrm{B} = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \cdot \begin{Vmatrix} -\eta\mu - 1 \\ -\mu \end{Vmatrix}^{\mathrm{T}} + \begin{Vmatrix} 1 + \alpha_1 + \alpha_2 \\ 1 - \alpha_1 + \alpha_2 \end{Vmatrix}^{\mathrm{T}} \cdot \begin{Vmatrix} 1 & \eta\mu^{-1} \\ 0 & \mu^{-1} \end{Vmatrix} \\ \mathrm{C} = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -\eta - \mu \end{Vmatrix} \cdot \frac{1}{2} \\ \mathrm{D} = \begin{Vmatrix} 1 + \beta_1 + \beta_2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -\eta - \mu \end{Vmatrix} \cdot \frac{1}{2} \cdot \begin{Vmatrix} -\eta\mu - 1 \\ -\mu \end{Vmatrix}^{\mathrm{T}} + \begin{Vmatrix} 1 & 0 \\ -2\eta & -1 \end{Vmatrix} \end{cases} \tag{4.9}$$

Furthermore, with

$$\vartheta = (1 + \beta_1 + \beta_2)(x - \mathrm{z}_1 - \mu(\eta \mathrm{z}_{1|n} + \mathrm{z}_{2|n})) \tag{4.10}$$

The system can be updated as shown in Table 4.2:

Table 4.2.
Optimised state space update of the eigenmorph form.

	y_n	$z_{1\|n+1}$	$z_{2\|n+1}$
Low Cut	$G \cdot \left(\vartheta + \frac{(1-\alpha_1+\alpha_2)(\eta z_{1\|n} + z_{2\|n})}{\mu}\right)$	$z_{1\|n} + \frac{\vartheta}{2}$	$-z_{2\|n} - (\eta+\mu)\cdot\frac{\vartheta}{2} - 2\eta z_{1\|n}$
Other	$G \cdot \left(\vartheta + \frac{(1-\alpha_1+\alpha_2)(\eta z_{1\|n} + z_{2\|n})}{\mu}\right) + z_{1\|n}$	$z_{1\|n} + \frac{\vartheta}{2}$	$-z_{2\|n} - (\eta+\mu)\cdot\frac{\vartheta}{2} - 2\eta z_{1\|n}$

We shall, of course, appreciate the simplicity and symmetry in Figure 4.1 and Figure 4.2. We also note that some gain blocks can be simplified. For instance

$$1 + \beta_1 + \beta_2 = \frac{4\pi^2}{\pi^2 + \nu + \pi\sqrt{2}\sqrt{\nu + \kappa}} \tag{4.11}$$

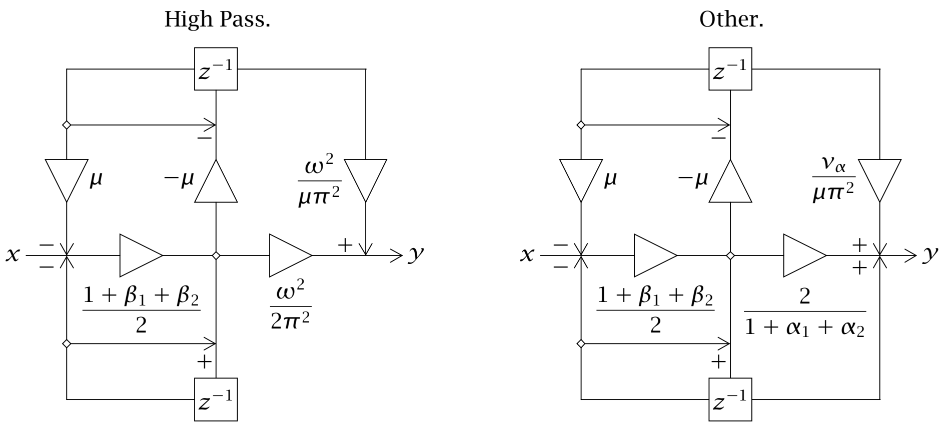


Figure 4.1. Optimised state variable form.

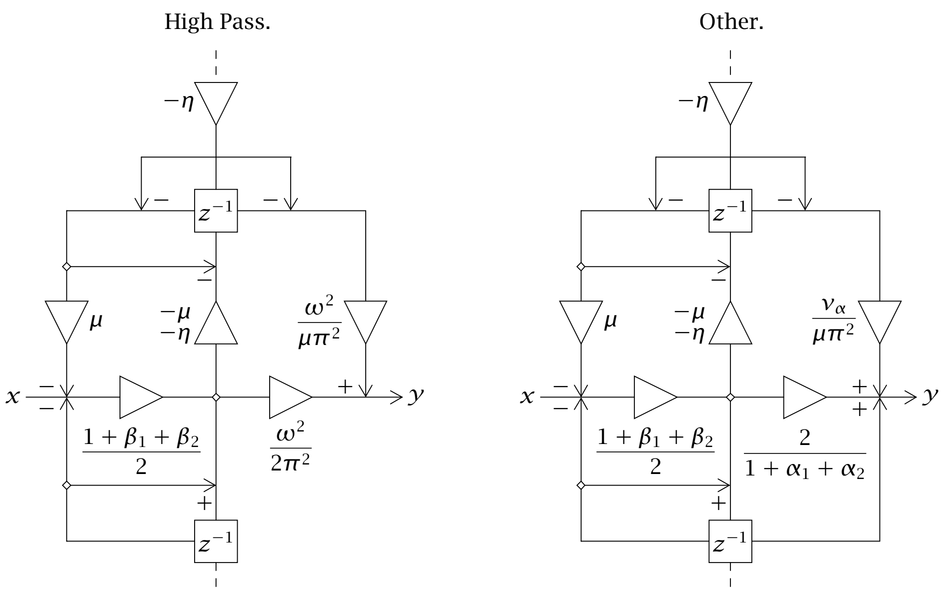


Figure 4.2. Optimised eigenmorph form.

4.4. NORMAL FORM

Normal matrices are of a great use in mathematics, due to their symmetric property. It has an orthonormal basis of eigenvectors. In other words, for a second order system, the states are perfectly 90° phase shifted, guaranteeing stability; unlike the Direct Form, whose states are phase shifted by 1 sample, as shown in Figure 5.2. Indeed, with

$$Q = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} \tag{4.12}$$

the state space can directly be built from the eigenvalues λ_1 and λ_2:

$$\begin{cases} A = \begin{Vmatrix} (\lambda_1 - 1)(\lambda_2 - 1) \end{Vmatrix} \\ B = \begin{Vmatrix} (\lambda_1 - 1)(\lambda_2 - 1) \end{Vmatrix} \cdot \begin{Vmatrix} (\lambda_1^2 + \lambda_1\alpha_1 + \alpha_2)(1 - \lambda_2) \\ (\lambda_2^2 + \lambda_2\alpha_1 + \alpha_2)(\lambda_1 - 1) \end{Vmatrix}^T \\ C = \begin{Vmatrix} 1 - \lambda_1 \\ 1 - \lambda_2 \end{Vmatrix} \\ D = \begin{Vmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{Vmatrix} + \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \end{cases} \tag{4.13}$$

REFERENCE

[1] Yuriy Ivantsov, “On the ideal bilinear and biquadratic digital filter.” [Online]. Available: https://ivantsovy.com/research/paper1.pdf

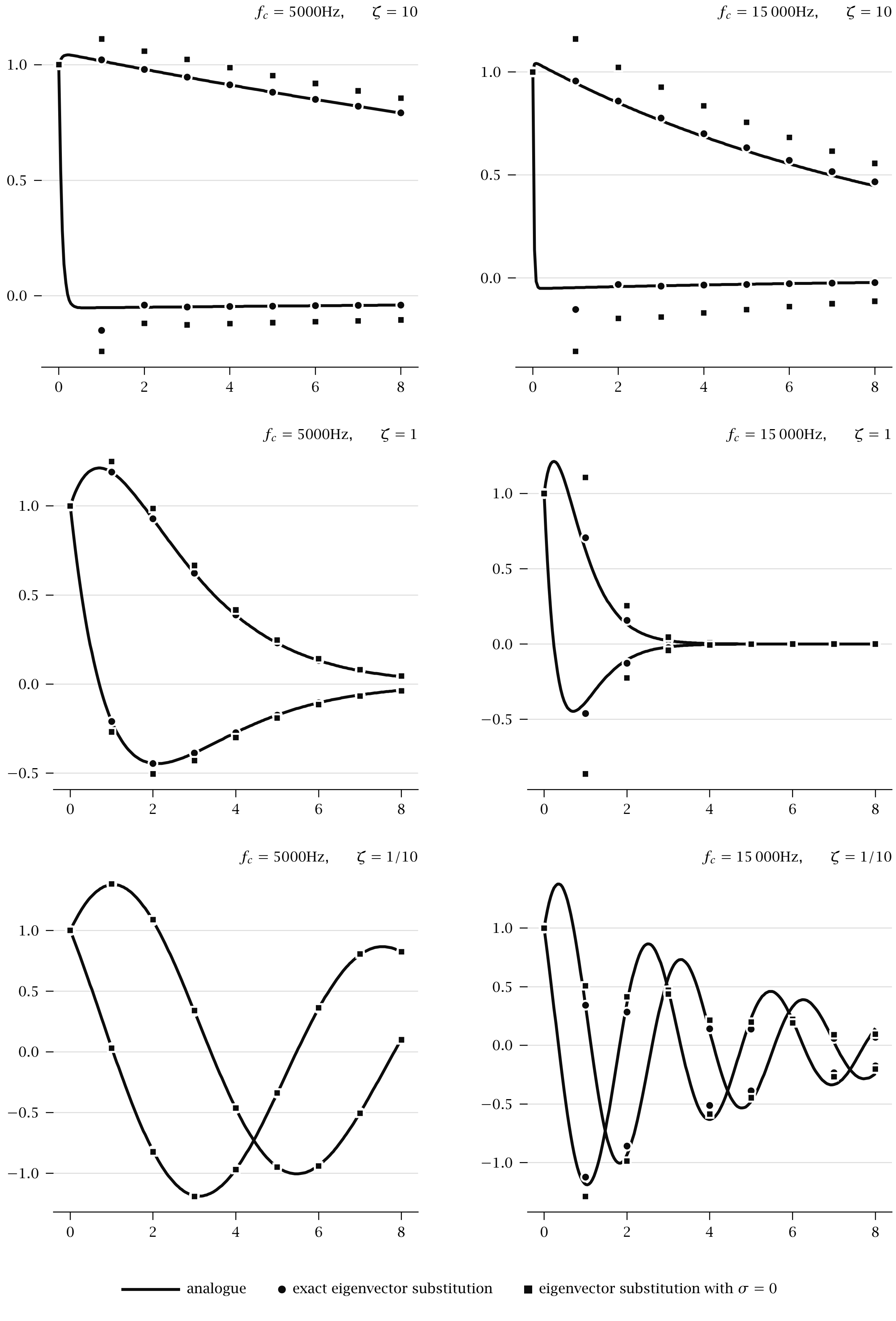


Figure 5.1. Comparison of the state evolution of the state variable form, based on the eigenvector discretisation.
$f_s = 44\,100\text{Hz}, \quad \sigma = \sqrt{2/3}\pi, \quad z_0 = \|1 \quad 1\|^{\mathrm{T}}$

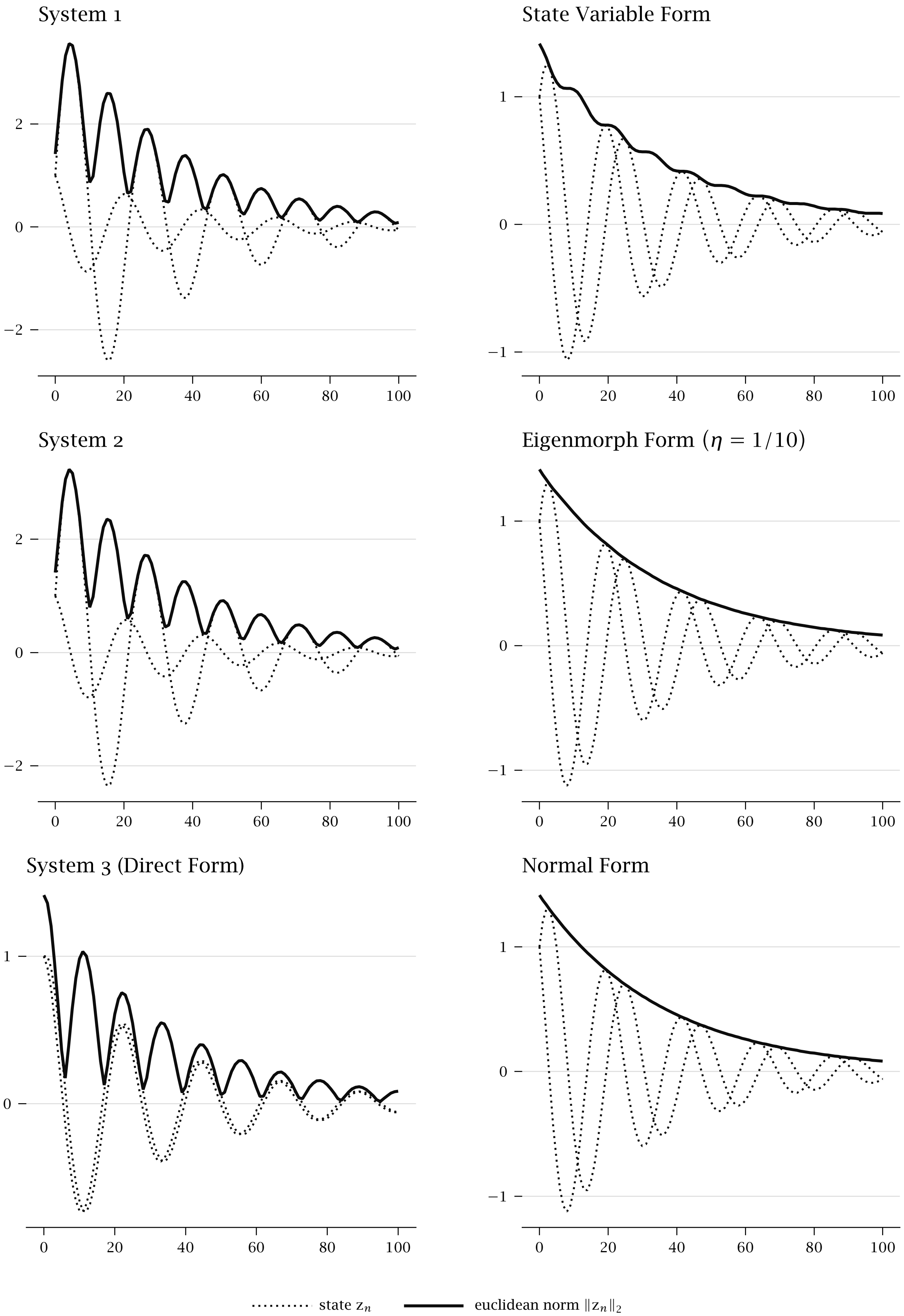


Figure 5.2. State evolution of various structures. On the left, the cause of potential instability; humps.
$f_c = 2000\text{Hz}, \quad f_s = 44\,100\text{Hz}, \quad \zeta = 1/10, \quad \sigma = \sqrt{2/3}\pi, \quad z_0 = \|1 \quad 1\|^{\mathrm{T}}.$